M2 Internship Report
Structure of Even-Hole-Free Graphs

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Abstract

An even-hole-free graph is a graph that has no induced even cycles of length at least 4. This class is structurally similar to the class of perfect graphs, which was one initial motivation for their study. But it is also of independent interest due to its relationship to β-perfect graphs. The class of even-hole-free graph is an object of much interest. A decomposition theorem and a recognition algorithm are known for this class, but the problem of coloring them in polynomial time or finding a maximum stable set are open in general. A natural approach is to study more closely the structure of graphs in this class, for instance by looking for a decomposition theorem or computing its treewidth. In the case where triangles are excluded, there exists a fruitful decomposition theorem by Conforti, Cornuéjols, Kapoor, and Vŭsković for even-hole-free graphs, which leads to small treewidth of graphs in this class. Furthermore, Adler, et al. prove that the class of (diamond, even-hole)-free graphs have unbounded rank-width which yields unbounded treewidth. For a larger subclass of even-hole-free graphs, for instance in (K_4, even-hole)-free class, such result is not known. The goal of this internship is to study this class of graphs.

At first, we conjectured that (K_4, even-hole)-free graphs have a small treewidth, but now we believe that the treewidth is unbounded, because we prove this for a class of graphs that has a similar but simpler structure, namely (K_4, even-hole)-free graphs. This class of graphs have been studied earlier. For instance, Radovanovic and Vŭsković prove that graphs in this class are 3-colorable, and there exists an algorithm that colors them in polynomial time. In this report, we prove that a $(\triangle, 3PC(\cdot,\cdot))$-free graphs may have arbitrarily large treewidth, by providing a construction of graphs that have such property. For further study, we aim to generalize our construction to the class of (K_4, even-hole)-free graphs.

1 Introduction

Throughout the report, all graphs are finite, simple, and undirected. The readers should refer to Appendix A for more notions and terms of Graph Theory that are used in this report. For a graph $G(V, E)$ (or G for short), $V(G)$ and $E(G)$ denote the set of vertices and edges of G respectively. When the context is clear, we will write V and E instead of $V(G)$ and $E(G)$. A *subgraph* H of G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is called an *induced subgraph* if $E(H)$ is formed by all possible connections of $V(H)$ in G. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G, and G is *F-free* if it does not contain F. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is $\mathcal{F}$-free if it is F-free, for every $F \in \mathcal{F}$. It is well-known that for any hereditary graph class $\mathcal{G}$ there is a family $\mathcal{F}$ of graphs s.t. $\mathcal{G}$ is $\mathcal{F}$-free.

One of the most interesting class of graphs that is $\mathcal{F}$-free is the class of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, we have $\chi(H) = \omega(H)$. Here, $\chi(H)$ denotes the *chromatic number* of H, i.e. the minimum number of colors needed to *properly* color the vertices of H (meaning that any pair of vertices that induces an edge in G are given different colors), and $\omega(G)$ is the size of largest clique in G, where a clique in a graph is an induced subgraph in which every pair of vertices are adjacent. The famous Strong Perfect Graph Theorem, conjectured by Berge (1961) and proved by Chudnovsky, Robertson, Seymour, and Thomas [6] states that a graph is perfect if and only if it does not contain an *odd hole* nor an *odd antihole*, where a hole is a chordless cycle of length at least four and an antihole is a hole in $\bar{G}$, with $\bar{G}$ is the *complement* of G (see Appendix A for the definition). A hole is *odd* or *even* depending on the parity of its length.

The class of even-hole-free graphs (or abbreviated to be EHF graphs) is structurally quite similar to the class of perfect graphs. The tools developped to study EHF graphs became a key of studying this class. Its structure was first studied by Conforti, Cornuéjols, Kapoor, and Vŭsković in [12], where they present a decomposition theorem for this class using 2-joins and k-star, that was used later in constructing poly-time recognition algorithm. Conforti, et al. [15] proved the Strong Perfect Graph Conjecture for 4-hole-free graphs, by decomposing Berge graphs using star cutsets and 2-joins into bipartite graphs and line graphs of bipartite graphs, and later, Chudsnovky, et al. use the similar approach to prove the general case, i.e. when 4-holes are allowed. We will describe the structure of EHF graphs in more details in Section 1.1.2.

The class of EHF graphs has taken a lot of attention, for a survey, see [27]. The study of this topic is also motivated by their connection to β-perfect graphs that was introduced by Markossian, et al. [21]. For a graph G, define $\beta(G) = \max\{\delta_H + 1\}$ where H is an induced subgraph of G, and δ_G is the minimum degree of a vertex in G. Consider a total ordering of vertices of G by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order shows that $\chi(G) \leq \beta(G)$, and we say G is *β-perfect* if $\chi(H) = \beta(H)$ for any induced subgraph H of G. It is easy to see that β-perfect graphs are even-hole-free (note that $\beta(C_{2k}) = 3$ and $\chi(C_{2k}) = 2$ for $k \geq 2$). Indeed, Markossian, et al. [21] show that G (and $\bar{G}$) is β-perfect if and only if it does not contains an even hole nor an even antihole, which is an interesting analogue of the Strong Perfect Graph Theorem.

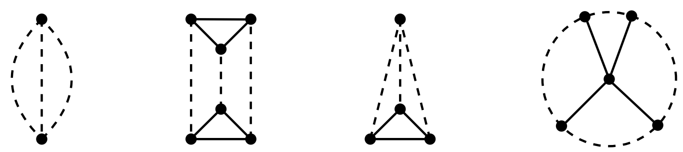
In the view of combinatorial optimization problems, as we know, for the class of perfect graphs, a lot of combinatorial problems such as maximum clique, maximum stable set, and optimal coloring can be solved in polynomial time. For EHF graphs, the complexities of finding a maximum stable set or an optimal coloring are not known. Fortunately, the maximum clique problem is polynomially solvable, because the class of 4-hole free graphs, which is a superclass of EHF graphs, have $\mathcal{O}(n^2)$ maximal cliques, and hence one can list all of them in polynomial time, precisely in $\mathcal{O}(n^{4.376})$ (see [27]).

For optimal coloring problem, in [1], it is proved that EHF graphs are χ-bounded, namely there exists a function f s.t. for every induced subgraph H of an EHF graph G, $\chi(H) \leq f(\omega(H))$ (in this case, $\chi(H) \leq 2\omega(H) - 1$ for every H). Furthermore, as been mentioned earlier, the class of β-perfect graphs, which is a subclass of EHF graphs can be colored in poly-time. Unfortunately, the poly-time recognition algorithm for this class is not known in general. Kloks, et al. [18] show that EHF graphs having no diamond are β-perfect, where the *diamond* is the graph obtained by removing one edge from a complete graph on 4 vertices. This motivates us to asking what kind of structural tools might give new insights for EHF graphs.

1.1 Background

1.1.1 Truemper Configurations

Truemper configurations are an important tool to describe the structures of several classes of graphs. Much research has been done in this area, for a survey, see [28]. We now define the Truemper configurations, which consist of *3-path configurations* and *wheels*, where 3-path-configurations are graphs isomorphic to a *prism*, a *pyramid*, or a *theta*, as depicted in the first three graphs of Fig. 1. These structures appear in a theorem of Truemper [26] that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities (see Corollary 1.1 of [26] or Theorem 2.1 of [27]).



(a) 3PC$(\cdot,\cdot)$ (b) 3PC$(\triangle,\triangle)$ (c) 3PC$(\triangle,\cdot)$ (d) wheel

FIGURE 1: Truemper configurations (solid lines represent edges and dashed lines represent chordless paths of length as described in the definition below)

Let x, y be two distinct vertices of G. A $3PC(x, y)$ is a graph induced by three chordless (x, y)-paths (which are denoted by P_1, P_2, and P_3), s.t. P_1, P_2, and P_3 all have length at least 2. Note that the $3PC(\cdot,\cdot)$ with fewest number of vertices is isomorphic to $K_{2,3}$, where $K_{2,3}$ is complete bipartite graphs where each component contains 2 and 3 vertices respectively. We say a graph G contains a $3PC(\cdot,\cdot)$ if it contains a $3PC(x, y)$ for some $x, y \in V(G)$. $3PC(\cdot,\cdot)$'s are also known as *thetas*.

Let $x_1, x_2, x_3, y_1, y_2, y_3$ be six distinct vertices of G s.t. $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce triangles. A $3PC(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths P_1, P_2, P_3 each connecting (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively, s.t. P_1, P_2, P_3 all have length at least 1. We say that a graph G contains $3PC(\triangle,\triangle)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. $3PC(\triangle,\triangle)$'s are also known as *prisms*.

Let x_1, x_2, x_3, x be four distinct vertices of G s.t. $\{x_1, x_2, x_3\}$ induces a triangle. A $3PC(x_1x_2x_3, x)$ is a graph induced by three chordless paths P_1, P_2, P_3 each connecting (x, x_1), (x, x_2), and (x, x_3) respectively, s.t. two of P_1, P_2, P_3 are of length at least 2, and the other has length at least 1. We say that a graph G contains $3PC(\triangle,\cdot)$ if it contains a $3PC(x_1x_2x_3, x)$ for some $x_1, x_2, x_3, x \in V(G)$. $3PC(\triangle,\cdot)$'s are also known as *pyramids*.

Recall that a hole in a graph is a chordless cycle of length at least 4. The constraints on the length of the paths in the three definitions above are designed in such a way that the union of any two of the paths in a $3PC$ induce a hole. Let us now define wheels. A *wheel*, denoted by (H, v) is a graph formed from a hole H and a vertex $v \in G \backslash H$ having neighbors $v_1, \cdots, v_n$ in H where $n \geq 3$. The hole H is called *rim*, the vertex v is the *center* of the wheel, and edges vv_i for $i \in \{1, \cdots, n\}$ are called *spokes* of the wheel. A subpath of H connecting v_i and v_{i+1} is called a *sector* of (H, v), and it is denoted by V_i. For a wheel (H, v), the set of neighbors of v in H is ordered from the lowest index to the highest in a counter-clockwise order.

The Truemper configurations play a key role in structural graph theory. It is easy to observe that a $3PC(\triangle, \cdot)$ contains an odd hole, and similarly $3PC(\cdot, \cdot)$ and $3PC(\triangle, \triangle)$ contain even holes. To see this, note that each of the $3PC$'s contains three different paths which implies that at least two of them have same parity. These paths together with the triangles or vertices contained in the $3PC$ create an odd or even hole. Therefore, $3PC(\triangle, \cdot)$-free graphs form a superclass of perfect graphs, and $3PC(\triangle, \triangle)$-free graphs and $3PC(\cdot, \cdot)$-free graphs are superclass of the class of EHF graphs. These superclasses capture some essential feature of the classes where they are included. Hence, Truemper configurations are a kind of tool to study some hereditary classes of graphs, which is also interesting in their own right.

A graph excluding all the Truemper configurations is called *universally signable* graph. It has been proved that this class has a very nice structure. Indeed, Conforti, et al. [9] give a decomposition theorem saying that a universally signable graph is either a clique, or a chordless cycle, or has a clique cutset. A natural question to ask is, whether Truemper configurations can be recognized in polynomial time. This question arised in the search for an algorithm to recognize perfect graphs. Indeed, it has been proved that detecting $3PC(\triangle, \cdot)$ can be done in time $\mathcal{O}(n^9)$ [5], and detecting $3PC(\cdot, \cdot)$ is in time $\mathcal{O}(n^{11})$ [7]. While pyramids and thetas can be detected in poly-time, detecting $3PC(\triangle, \triangle)$'s and detecting wheels are NP-Complete problems (see [17], [20]).

1.1.2 Structure of even-hole-free graphs

A graph G is *connected* if for every pair of vertices $u, v \in V$, there exists a path from u to v. A graph that is not connected is sometimes called *disconnected*. In a connected graph G, a subset S of V is a *cutset* if the removal of $G[S]$ disconnects G. A *decomposition* of a graph G is a set of subgraphs $H_1, \cdots, H_k$ that partition $E(G)$, i.e. for every $1 \leq i, j \leq k$, $\bigcup_i H_i = G$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$. A *decomposition theorem* for a class of graphs $\mathcal{G}$ states that every graph in $\mathcal{G}$ either has a particular type of a cutset or belongs to a *basic* (i.e. undecomposable) subclass of $\mathcal{G}$. The following cutsets are used in the decomposition of EHF graphs (for diagram illustrating these cutsets, see Fig. 17 in Appendix).

A cutset S is a *clique cutset* if S induces a clique. S is a *k-star cutset* if it is comprised of a clique C of size k s.t. every vertex of $S \setminus C$ has at least one neighbor in C. A 1-star is also referred to as a *star*. If $S = N[C]$, then S is called a *full k-star*. A graph G has a *2-join* (V_1, V_2), with special sets (A_1, A_2, B_1, B_2), if the vertices of G can be partitioned into sets V_1 and V_2 s.t. the following hold.

(i) For $i = 1, 2$, $A_i \cup B_i \subseteq V_i$, and A_i and B_i are nonempty and disjoint

(ii) Every vertex of A_1 (resp. B_1) is adjacent to every vertex of A_2 (resp. B_2), and these are the only adjacency between V_1 and V_2.

(iii) For $i = 1, 2$, $G[V_i]$ contains a path with one end-node in A_i and the other in B_i. Moreover, $G[V_i]$ is not a chordless path.

A *long* $3PC(\triangle, \cdot)$ is a 3PC in which all the paths P_1, P_2, P_3 has length greater than 1. Note that when P_1, P_2, P_3 have a same parity, a long $3PC(\triangle, \cdot)$ is EHF, but have no k-star cutset nor a 2-join. Moreover, it is an obvious fact that cliques and an odd holes are EHF. It yields that cliques, odd holes, and long $3PC(\triangle, \cdot)$'s form a basic class of the class of EHF graphs. Furthermore, Conforti, et al. [12] introduced another basic class of EHF graphs that is called *nontrivial basic graph* (see Fig. 2).

In order to understand the construction, we need to introduce the notion of *line graph*. Given a graph G, its line graph $L(G)$ (or L for short) is a graph such that each vertex of L represents an edge of G, and for $p, q \in V(L)$, $pq \in E(L)$ if and only if their corresponding edges $e_p, e_q \in E(G)$ share a common end-node in G. Let L be the line graph of a tree. Note that every edge of L belongs to exactly one maximal clique and that every vertex of L belongs to at most two maximal cliques. The vertices of L that belong to exactly one maximal clique are called *leaf vertices*. A clique of L is *big* if it has size at least 3. In the graph obtained from L by removing all edges in big cliques, the connected components are chordless paths (possibly of length 0). Such a path P is an internal segment if it has its end-nodes in distinct big cliques (when P is of length 0, it is called an internal segment when the vertex of P belongs to two big cliques). The other paths P are called *leaf segments*. Note that one of the end-nodes of a leaf segment is a leaf vertex.

A nontrivial basic graph R is defined as follows: R contains two adjacent vertices x and y, called the *special* vertices. The graph L induced by $R \backslash \{x, y\}$ is the line graph of a tree and contains at least two big cliques. In R, each leaf vertex of L is adjacent to exactly one of the two special vertices, and no other vertex of L is adjacent to special vertices. The last condition for R is that no two leaf segments of L with leaf vertices adjacent to the same special vertex have their other end-node in the same big clique. The internal segments of R are the internal segment of L.

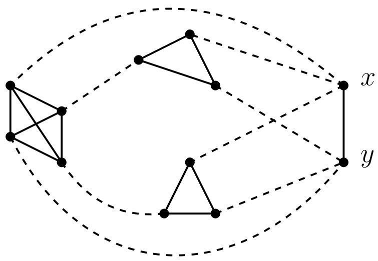


FIGURE 2: Nontrivial Basic Graph of EHF Class

Theorem 1.1. *(Conforti, Cornuéjols, Kapoor, and Vŭsković [12]) A connected 4-hole-free, odd-signable graph is either a clique, a hole, a long $3PC(\triangle, \cdot)$ or a nontrivial basic graph, or it has a 2-join or k-star cutset, for $k \leq 3$.*

This decomposition theorem is the main tool to construct the first known polynomial time recognition algorithm for EHF graphs. Moreover, such recognition algorithm for this class can be used to find an even hole in a graph G, if one exists. Later, da Silva and Vŭsković [16] strengthen this decomposition theorem by decomposing the class of EHF graphs with the star-cutsets and 2-joins. The problem with the star cutset is that it can be very big, so there is not much structure one can work with. In the other hand, 2-joins have a better structure within the cutset. In the class of EHF and Berge graphs, there is an interesting relationship between star cutset decomposition and 2-join decomposition, where one can build a decomposition tree by first doing star cutset decompositions and then 2-join decompositions, with leaves are undecomposable blocks (i.e. the basic graphs in the class). Analogous separation exists in Berge graphs, using *skew cutsets* (see Appendix A for the definition) and 2-joins [25].

1.2 Problems and Outline

A *triangle*, denoted by K_3 or $\triangle$ is a complete graph on three vertices. Conforti, Cornuéjols, Kapoor, and Vŭsković [11] prove that when triangles are excluded, the class of EHF graphs has a very simple structure, which implies that it has a small treewidth [4] (see Section 2 for more details), in particular for a $\triangle$-free, odd signable graph G, $tw(G) \leq 5$. A very natural question to ask is, whether the similar properties still hold if triangles are allowed. Kloks, Muller, and Vŭsković [18] has investigated the structure of EHF graphs that do not contain *diamonds*, where a diamond is a graphs that is obtained by removing an edge of a K_4, the complete graph on 4 vertices. They prove that every graph in this class contains a *simplicial extreme* (i.e., a vertex that is either of degree 2 or whose neighborhood induces a clique). This characterization implies that for every diamond-free EHF graph G, $\chi(G) \leq \omega(G) + 1$, which means that this class belongs to the χ-bounded family of graphs (a class of graph $\mathcal{G}$ is said to be χ-bounded if there is a function f s.t. $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$). Moreover, the existence of simplicial extremes also shows that diamond-free EHF graphs are β-perfect.

As the first approach, we investigate the class of EHF graphs that do not contain a K_4. Therefore, the main question of this research is, whether the class of K_4-free EHF graphs has a simple structure, for example by checking if the treewidth of is bounded. In fact, while the class of $\triangle$-free, odd-signable graphs has a quite simple structure, it turns out that the structure of the class of K_4-free EHF graphs is more complicated. In this class, wheels interact in a more complex way. For instance, in the class of $\triangle$-free, odd-signable graphs, it is not possible to have wheels with a same rim, but with two different centers that are adjacent, meanwhile this case is possible in the class of K_4-free EHF graphs.

Since the class of K_4-free EHF graphs were more complicated than expected, we decided to study the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs, that are simpler, but have a kind of similar structure for wheels, namely a graphs in this class also may contains wheels of two adjacent centers having a same rim. Note that Radovanovic and Vŭsković [22] has analyzed some properties of graph that belongs to the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs. They show that this class is indeed 3-colorable, and there exists an algorithm that color them in a quadratic time. Our interest is to study the structure of $(\triangle, 3PC(\cdot,\cdot))$-free graphs, and computing the treewidth of graphs in this class. For the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs, we conjectured that the treewidth is bounded, but this is not the case. A graph in this class may have arbitrarily large treewidth (see Section 3.2). Indeed, there exists graphs that contain a complete graph of arbitrarily size as a minor, which implies that the treewidth is huge.

This report is organized as follows. We have already seen Section 1. In Section 2, we discuss the class of $\triangle$-free EHF graphs. We present the decomposition and characterization of the class of $\triangle$-free, odd-signable graphs, and we give the proof that the class of $\triangle$-free, odd-signable graphs and the class of $\triangle$-free EHF graphs are indeed, homeomorphic (see Theorem 2.4). Section 3 discusses the characterization of the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs. We prove that this class of graphs has unbounded treewidth, by providing $(\triangle, 3PC(\cdot,\cdot))$-free graph that has arbitrarily large treewidth. In Section 4, we discuss the class of K_4-free EHF graphs. We try to generalize the construction that we describe in proving the large treewidth of $(\triangle, 3PC(\cdot,\cdot))$-free graphs, to see whether the class of K_4-free EHF graphs also have unbounded treewidth. In the end, the Appendix provides some background of Graph Theory and some proofs that cannot be covered in the main sections.

1.3 Terminologies and Notations

Let G be a graph, and $x \in V \backslash U$ with $U \subsetneq V$. We say vertex x is adjacent to U, if x is adjacent to some vertex of U, and it is *strongly adjacent* to U, if it is adjacent to at least two vertices of U. For $S \subsetneq V(G)$, we denote by $G[S]$ the subgraph of G induced by the subset of vertices S. A path P is a sequence of distinct vertices $p_1, \cdots, p_k$, $k \geq 1$, s.t. $p_i p_{i+1} \in E(G)$, for all $1 \leq i < k$. If $p_1 = p_k$ then we call it a *cycle*. Vertices p_1 and p_k are called the *end-nodes* of the path P. Subpath $P \backslash \{p_1, p_k\}$ is denoted by $\mathring{P}$, and the set of vertices of $\mathring{P}$ is called *internal* vertices of P. Let p_i and p_j be two vertices of P, s.t. $j \geq i$. The path $p_i, p_{i+1}, \cdots, p_j$ are called the (p_i, p_j)-subpath of P and it is denoted by $p_i P p_j$. The *length* of P is the number of edges in P, and it is denoted by $|P|$ (similar definition for cycle). An edge $e \in E(G)$ is a *chord* of a path P if $E(P)$ can be partitioned into two sets P_1 and P_2 such that both $P_1 \cup \{e\}$ and $P_2 \cup \{e\}$ are paths. A path (or a cycle) that has no chord is said to be *chordless*.

Along this report, in some figures, unless stated, solid lines represent edges and dashed lines represent chordless paths of length at least one. For a vertex $x \in V(G)$, $N_G(x)$ (or $N(x)$ when the context is clear) denotes the set of neighbors of x in G, and $N_G[x] = N_G(x) \cup \{x\}$. Similarly, for $U \subsetneq V(G)$, $N_G[U] = N_G(U) \cup U$. For simplicity, a singleton set $\{x\}$ will sometimes be denoted with just x, so $G \backslash x$ denotes the subgraph of G that is induced by $V(G) \backslash \{x\}$. Similarly, $G \backslash H$ denotes the subgraph induced by $V(G) \backslash V(H)$, for a subgraph H of G. When it is clear, for a subgraph H of G, we write H instead of $V(H)$ to denote the set of vertices of H.

2 The Class of △-free EHF graphs

A graph with no $3PC(\cdot,\cdot)$, $3PC(\triangle,\triangle)$, and even wheel is called *odd-signable* graph. This name comes from the Truemper's theorem [26], which implicitly relates the *signing* (see Appendix A for the definition) of a graph and the existence of Truemper configurations in the graph. We are particularly interested in this class of graphs because the class of odd-signable graphs is a superclass of EHF graphs.

2.1 Decomposition

Conforti, et al. [11] gives a characterization of △-free graphs that are odd-signable using a *clique-cutset decomposition* and *wheel decomposition*. Using this decomposition theorem, they build a poly-time recognition algorithm and a very fruitful construction of the class of △-free, odd-signable graphs (Theorem 2.3).

Definition 2.1. (Ear) *A chordless (x, z)-path P is an ear of the hole H if the internal vertices of P belong to $V(G) \backslash V(H)$, vertices x, z have a common neighbor y in H, and $(H \backslash y) \cup P$ induces a hole H'. We say that vertex y is the center of the ear P, vertices x, z are the attachments of P in H, and that H' is obtained by augmenting H and P.*

A graph G is said to be obtained from a graph G' by an *ear addition* if the vertices of $G \backslash G'$ are the internal vertices of an ear of some hole H in G'. Conforti, et al. [11] describes a procedure to construct all △-free, odd-signable graphs. In their construction, every connected △-free, odd-signable graph which does not contain a K_1, K_2 cutset is either a cube, or it can be obtained starting with a hole, by a sequence of *good ear addition*, where the *cube* is a graph obtained from the complete bipartite graph $K_{4,4}$ by removing a perfect matching (i.e. an independent set of $E(G)$ in which every vertex of the graph is incident to exactly one edge of the set).

Definition 2.2. (Good Ear Addition) *Let G be obtained from G' by adding an ear P with attachments $x, z \in V(G')$. It is a good ear addition if $|N_P(y)|$ is odd, and*

(i) G' contains no wheel (H, v) s.t. $x, y, z \in V(H)$ and $vy \in E(G)$;

(ii) G' contains no wheel (H, y) s.t. $x, z \in V(H)$.

Theorem 2.3. (Theorem 6.4 in [11]) *Let G be a connected $\triangle$-free graph with $|V(G)| \geq 3$, s.t. G is not a cube, and it contains no K_1 or K_2-cutset. Then, G is odd-signable if and only if G can be obtained, starting from a hole, by a sequence of good ear additions.*

The two forbidden wheels in a $\triangle$-free, odd-signable graphs as stated in Definition 2.2 are called wheel of *type-1* and *type-2* respectively. A *graph subdivision* $\tilde{G}$ of a graph G is a graph obtained from a subdivision of edges in G. The *subdivision* of some edge $uv \in E(G)$ returns a graph $\tilde{G}$ containing one new vertex w, and with an edge set replacing uv by two new edges, uw and wv. We say that G and $\tilde{G}$ are *homeomorphic*.

In most papers about EHF graphs, the authors in fact study a more general class of odd-signable graphs (for instance, C_4-free, odd-signable graphs). This raises a question: could it be that EHF graphs are structurally simpler than odd-signable graphs? This could be helpful if the answer is yes. In the following, we show that the class of $\triangle$-free EHF graphs is homeomorphic to the class of $\triangle$-free, odd-signable graphs when the cube is excluded (Theorem 2.4). This theorem gives a negative answer for the above question, in the ($\triangle$, cube)-free case, since both graphs are indeed homeomorphic. Note that, excluding cubes is necessary, since a cube contains a C_4, and any subdivision of a cube always contains $3PC(\cdot,\cdot)$, implying that it contains an even hole.

Theorem 2.4. *Let G be a $\triangle$-free, odd-signable graph containing no cube and no K_1, K_2 cutset. Then G can be subdivided in such a way that the new graph $\tilde{G}$ is a $\triangle$-free EHF graph containing no cube nor K_1, K_2 cutset.*

Proof. Let G be a $\triangle$-free, odd-signable graph containing no cube and no K_1, K_2 cutset. By Theorem 2.3, G is obtained starting from a hole, by a sequence of good ear additions. So G is obtained from G' by adding a good ear P, where G is also a graph of the same class as G. Let H' be a hole of G' in which P is attached, and x, z be the vertices where the two end-nodes t and t' of P are attached. Moreover, let $t_1, \cdots, t_n$ be a sequence of vertices in $P \backslash \{t_1, t_n\}$ (appear in counterclockwise order along P) that are adjacent to y (see Fig. 3). By the definition of good ear addition, n is odd and there is no edge connecting a vertex of $\mathring{P}$ to a vertex of $G' \backslash \{y\}$. We will prove that if G' is an EHF graph, then so is G.

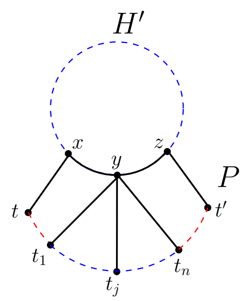


FIGURE 3: A good ear P attached to H' (the red and blue dashed edges represent paths of length at least 1 and 2 respectively)

Note that a hole of G is either a hole of G', or it is one of (x, tPt_1, y, x), (y, t_nPt', z, y), (y, t_jPt_{j+1}, y) for $1 \leq j \leq n-1$ (let us call them *trivial holes*), or it is obtained by augmenting P with a chordless (x, z) path in $G' \backslash \{y\}$ (let us call them *nontrivial*). We aim to *kill* all even holes in G. By killing an even hole, we mean subdividing one of its edges s.t. its length is no more even, but preserving the parity of other odd holes in the graph. We will prove that we can inductively kill all even holes in G.

If G is a hole, then we can easily kill it if its length is even, by subdividing one of its edges. For induction, assume that G' is even-hole-free. First, we can kill all trivial holes in G by subdiving some edges of P in such a way that all the trivial holes in G have odd length. Let us denote the graph obtained by doing these subdivision $\tilde{G}$. Note that there are an even number of trivial holes in G, because y has an odd number of neighbors in P. Moreover, $|tPt_1|$ and $|t_nPt'|$ are both of even length, and for $1 \leq j < n$, $|t_jPt_{j+1}|$ is odd. Hence, the length of P is even, i.e. it contains an odd number of vertices.

Furthermore, we are going to prove that all the nontrivial holes in $\tilde{G}$ are indeed of odd length. For a contradiction, suppose that C is a nontrivial hole contained in $\tilde{G}$ and it has an even length. Denote $Q = C \backslash P$, which is a path connecting x and z in $G' \backslash y$. First, suppose that $C = H$ (recall that H is the hole obtained by augmenting $H' \backslash y$ and P). Note that Q is an (x, z)-path in $H' \backslash y$. Moreover, since $|P|$ is even, then $|Q|$ must be also even. But then, $|H'| = |yxQzy|$ is also even, contradicts our assumption that G' is even-hole-free.

Now suppose that $C \neq H$. If y has no neighbor in Q, then $C' = yxQzy$ is a hole in G', and thus by induction hypothesis it has an odd length, i.e. it contains an odd number of vertices. It yields that $Q = C' \backslash y$ contains an even number of vertices, i.e. $|Q|$ is odd, and hence $C = xQzt't_nPt_1tx$ contains an odd number of vertices, a contradiction. So assume that y is adjacent to some vertices of Q. Note that (C, y) is a wheel in G. By Theorem 2.3, $\tilde{G}$ does not contain even wheel. Thus, y certainly has an even number of neighbors in $Q \backslash \{x, z\}$, because y has an odd number of neighbors in P. This implies that Q contains an odd number of sectors of (C, y). By the induction hypothesis, each of the sectors must have odd length. But then the length of Q is odd, implying that $|C| = |P| + |Q|$ is odd, a contradiction. This completes our induction.

So $\tilde{G}$ is ($\triangle$, even-hole)-free. Moreover, since subdividing edges does not create K_1, K_2 cutset nor a cube, then $\tilde{G}$ has no K_1, K_2 cutset nor a cube.

□

Open question. We are questioning if the above theorem true in general, namely whether the odd-signable graphs are homeomorphic to EHF graphs when cubes or squares (i.e. cycles of length 4, or C_4) are excluded. If the answer is negative, then what kind of structures that are not one of the Truemper configurations that implies the presence of even hole (like a cube for example)?

2.2 Small Treewidth

We now introduce the notion of a *tree decomposition* and *treewidth* of a graph. This notion was introduced by Robertson and Seymour in their work on graph minors [23]. It is one of powerful tools for many graph algorithmic studies. A lot of problems, which are NP-Complete in general, can be solved in polynomial time even in linear time for graphs with small treewidth. In general, computing the treewidth of a graph is NP-Hard. In some classes of graphs, such as the class of chordal graphs, cographs, circle graphs, and distance hereditary graphs, treewidth can be computed in polynomial time. Bodlaender [3] constructed a linear time algorithm that finds an optimal tree decomposition for a graph with bounded treewidth.

Definition 2.5. *A tree decomposition of a graph $G(V, E)$ is a tree $T(I, F)$, where:*

(i) each vertex $i \in I(T)$ is labeled by a subset $B_i \subseteq V(G)$, satisfying $\bigcup_{i \in I} B_i = V(G)$;

(ii) for any edge $uv \in E(G)$, there exists an $i \in I(T)$ with $u, v \in B_i$;

(iii) for any $v \in V(G)$, the subgraph of T induced by $\bigcup_{v \in B_i} i$ forms a subtree of T.

Remark 2.5.1. *For any i, B_i is referred to as a bag, and the number of vertices contained in B_i defines the size of the bag.*

Every graph has a trivial tree decomposition for which T has one vertex including all of $V(G)$. This partly motivates the following definition.

Definition 2.6. *The treewidth of G is the minimum integer k such that there exists a tree decomposition G with every bag has size at most $k + 1$. We denote the treewidth of G by $tw(G)$.*

Using the construction based on Theorem 2.3, Cameron, et al. [4] proves that the class of $\triangle$-free, odd-signable graphs have treewidth at most 5, by some chordalization technique. Later they use this result to show that the class of (cap, C_4)-free, EHF graphs has treewidth at most $6\omega(G) - 1$ (a cap is a hole with exactly one chord yielding a triangle). As been mentioned earlier, this shows that optimal coloring and maximum stable set problems can be solved in poly-time in the class of $\triangle$-free, EHF graphs.

3 The Class of $(\triangle, 3PC(\cdot, \cdot))$-Free Graphs

Our original motivation of discussing the class of $(\triangle, 3PC(\cdot, \cdot))$-free graphs is to have an insight about the treewidth of the class of K_4-free EHF graphs, that seems to be structurally similar but more complicated.

3.1 Wheels in the class of $(\triangle, 3PC(\cdot, \cdot))$-free graphs

Wheels in a $(\triangle, 3PC(\cdot, \cdot))$-free graph interacts in a more complicated way compared to a $\triangle$-free, odd signable graph. Indeed, it may contains a wheel whose centers induce a graph on at least two vertices (for instance, Fig. 4 shows a wheel whose centers induce a path and a hole respectively). Let us call it *multiple wheel* and denote it by (H, C). Note that it has a same rim H, and its center C is a $(\triangle, 3PC(\cdot, \cdot))$-free graph. A multiple wheel is called a *2-wheel* if C contains exactly two vertices. For $V(C) = \{u, v\}$, we denote the 2-wheel by $(H, \{u, v\})$. In this case, spokes that go from u (resp. v) are called *u-spokes* (resp. v-spokes). A sector of (H, u) (resp. (H, v)) is called *u-sector* (resp. v-sector), and a subpath $P = h_u H h_v$ of H where $h_u \in N_H(u)$ and $h_v \in N_H(v)$ that does not contain any other neighbor of u or v is called a *(u, v)-sector*.

Definition 3.1. *A 2-wheel $(H, \{u, v\})$ in a $(\triangle, 3PC(\cdot, \cdot))$-free graph G is called nested 2-wheel if it satisfies the following properties:*

(i) there is a unique u-sector that contains all neighbors of v;

(ii) there is a unique v-sector that contains all neighbors of u.

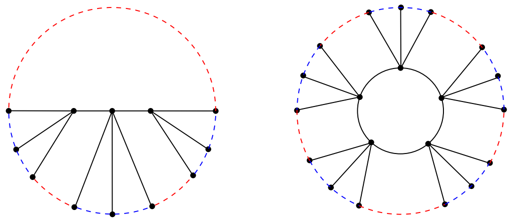


FIGURE 4: Wheel centered at a path or a hole (the red and blue dashed edges represent paths of length at least 1 and 2 respectively)

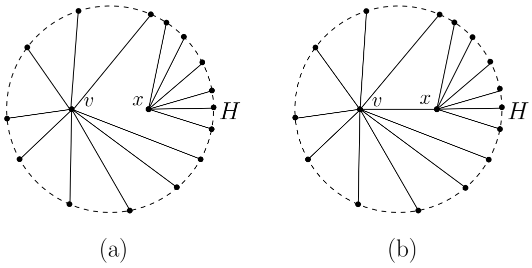


FIGURE 5: A nested 2-wheel centered at v and x

Remark 3.1.1. *When u and v are adjacent, then u and v have no common neighbor in H, because G is $\triangle$-free.*

Definition 3.2. *A 2-wheel $(H, \{u, v\})$ in a $(\triangle, 3PC(\cdot,\cdot))$-free graph G is called alternated 2-wheel if it satisfies the following properties:*

(i) u and v are adjacent;

(ii) every u-sector contains either ≥ 2 neighbors of v or none of them;

(iii) every v-sector contains either ≥ 2 neighbors of u or none of them;

(iv) there are at least four (u, v)-sectors.

Remark 3.2.1. *Note that $(H, \{u, v\})$ is an alternated 2-wheel only if both u and v have at least five neighbors in H.*

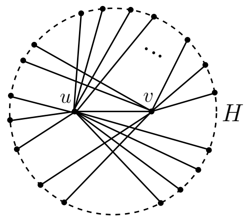


FIGURE 6: An alternated 2-wheel centered at uv

In the following, we give characterizations of 2-wheels in a $(\triangle, 3PC(\cdot,\cdot))$-free graph.

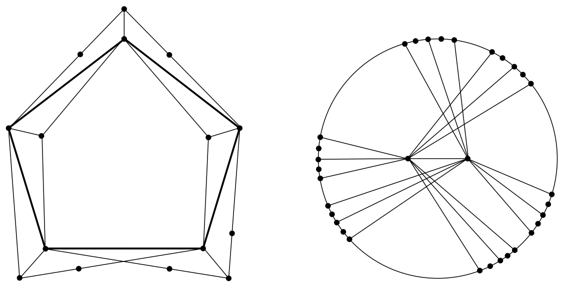
Lemma 3.3. *Let G be a $(\triangle, 3PC(\cdot,\cdot))$-free graph which does not contain a cube. Let (H, v) be a wheel contained in G, and x be a vertex in $G \backslash W$ that is strongly adjacent to (H, v). Then $H \cup \{v, x\}$ either induces a nested 2-wheel, or an alternated 2-wheel centered at $\{v, x\}$.*

Proof. See Appendix B.1.

3.2 Unbounded Treewidth

An undirected graph H is called a *minor* of the graph G if H can be formed from G by deleting edges and vertices, and by *contracting* edges, where an *edge contraction* is an operation which removes an edge from a graph while simultaneously merging the two vertices u, v that are previously joined by an edge, into a single vertex w, with $N_H(w) = N_G(u) \cup N_G(v)$. In the class $\mathcal{G}$ of $(\triangle, 3PC(\cdot,\cdot))$-free graph, note that a hole of minimum length is the minimum K_3-minor in $G \in \mathcal{G}$. It is an easy fact that a wheel with rim of length 6 and three spokes is a minimum subdivision of K_4 contained in G. Hence, if G contains a wheel then it contains a K_4-minor.

A *planar* graph is a graph that can be drawn in the plane with no two edges crossing. Kuratowski [19] states that a graph is planar if and only if it does not contain subdivision of K_5 nor of $K_{3,3}$ as a subgraph. In fact, a graph in $\mathcal{G}$ may contain a K_5-minor [22] (see Fig. 7 (a)). Moreover, an alternated 2-wheel may contain a K_5 as a minor (see Fig 7 (b)). So $\mathcal{G}$ contains some graphs that are not planar.



(a) (b)

FIGURE 7: A $(\triangle, 3PC(\cdot,\cdot))$-free graph that contains K_5 as a minor

Lemma 3.4. *(Robertson, Seymour [24]) Let H be a minor of G. Then $tw(H) \leq tw(G)$*

It is known that the treewidth of a complete graph K_l on l vertices is $l-1$. It is based on the fact that for a graph G, if a set $S \subseteq V(G)$ induces a clique in G and $T(I, F)$ is a tree decomposition of G, then there exists $i \in V(T)$ s.t. $S \subseteq B_i$. Hence, by Lemma 3.4, it follows that a graph G containing K_l as a minor has treewidth at least $l-1$. In the following section, we are going to show that a $(\triangle, 3PC(\cdot,\cdot))$-free graph may have arbitrarily large treewidth. The *girth* of a graph is the length of a shortest cycle contained in the graph. In the following section, for any integer $l \geq 1, k \geq 3$, we give a construction of $(\triangle, 3PC(\cdot,\cdot))$-free graph that contains a K_l as minor. See Fig. 10 for an example of graph described below.

Definition 3.5. *Let $l \geq 1$ and $k \geq 3$ be integers. An (l, k)-graph is any graph satisfying the following properties.*

1. *$V(G)$ is partitioned into l sets that induce paths $P_1, \dots, P_l$. So, $V(G) = V(P_1) \cup \dots \cup V(P_l)$. Such paths are sometimes called layers.*
2. *For every $1 \leq i \leq l$, $P_i = v_{i,1} \dots v_{i,p_i}$, where P_1 has length zero, i.e. $P_1 = v_{1,1}$.*
3. *For every $2 \leq i \leq l$ and every vertex v in P_i, v has at most one neighbor in $V(P_1) \cup \dots \cup V(P_{i-1})$. Such a neighbor (if any) is called the* ancestor *of v.*
4. *For every $1 \leq i \leq l-1$ and every vertex v in P_i, v has 3 or 6 neighbor in P_{i+1}. Moreover, v has neighbors in the paths $P_{i+2}, \dots, P_l$.*

5. *Let $1 \leq i < j \leq l$ be integers, and $v_{j,a}$ and $v_{j,b}$ be vertices in P_j with ancestors $v_{i,a'}$ and $v_{i,b'}$ in P_i. Then $a \leq b$ if and only if $a' \leq b'$.*

6. *Let $2 \leq i \leq l$ be an integer, and $v_{i,a}$, $v_{i,b}$ be two distinct vertices in P_i. If $v_{i,a}$, $v_{i,b}$ both have ancestors then $b - a \geq k - 3$. If $v_{i,a}$, $v_{i,b}$ have the same ancestor, then $b - a \geq k - 2$.*

7. *For all $1 \leq i \leq l-1$ and all $v \in V(P_i)$ with no ancestor, v has exactly three neighbors in P_{i+1}, namely $v_{i+1,a}$, $v_{i+1,b}$ and $v_{i+1,c}$ where a, b, c are integers satisfying $a < b < c$, $b - a = k - 2$ and $c - b = k - 2$ (see Fig. 8).*

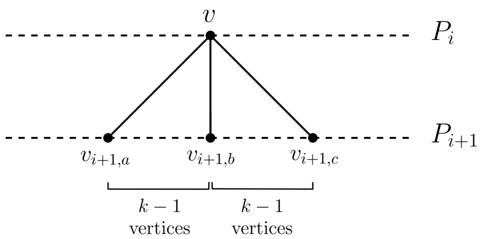


FIGURE 8: Property 7

8. *For all $1 \leq i \leq l-1$ and all $v \in V(P_i)$ with an ancestor w, v has exactly six neighbors in P_{i+1}, namely $v_{i+1,a}$, $v_{i+1,b}$, $v_{i+1,c}$ $v_{i+1,a'}$, $v_{i+1,b'}$, $v_{i+1,c'}$ where a, b, c, a', b', c' are integers satisfying $a < b < c < a' < b' < c'$, and $b - a, c - b, b' - a', c' - b' \geq k - 2$. Moreover, w has exactly three neighbors in $v_{i+1,c}P_{i+1}v_{i+1,a'}$, namely $v_{i+1,a''}$, $v_{i+1,b''}$, $v_{i+1,c''}$ where a'', b'' and c'' are integers satisfying $c < a'' < b'' < c'' < a'$, and $a'' - c, b'' - a'', c'' - b'', a' - c'' = k - 2$ (see Fig. 9).*

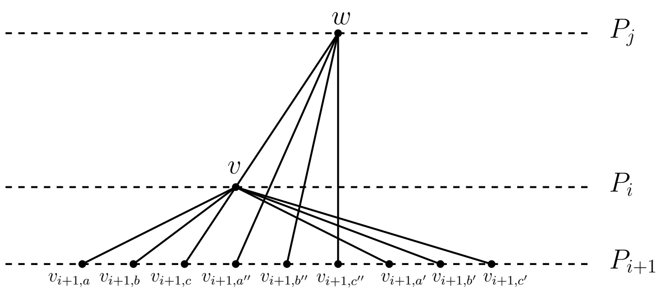


FIGURE 9: Property 8

9. *There are no other edges than the ones specified above.*

We denote the graph by $G_{l,k}$.

Remark 3.5.1. *For a graph $G_{l,k}$ described above,*

(i) An index of a vertex is the index of the path in which the vertex is contained. For instance, every vertex $v \in P_i$ has index i for $1 \leq i \leq l$.

(ii) For $2 \leq i \leq l$, and $j < i$ with $j \neq 0$, a vertex $v \in P_i$ is said to be of type-j if it has an ancestor in P_{i-j}. If v has no ancestor in $P_1 \cup \cdots \cup P_{i-1}$, then we say that it is of type-0.

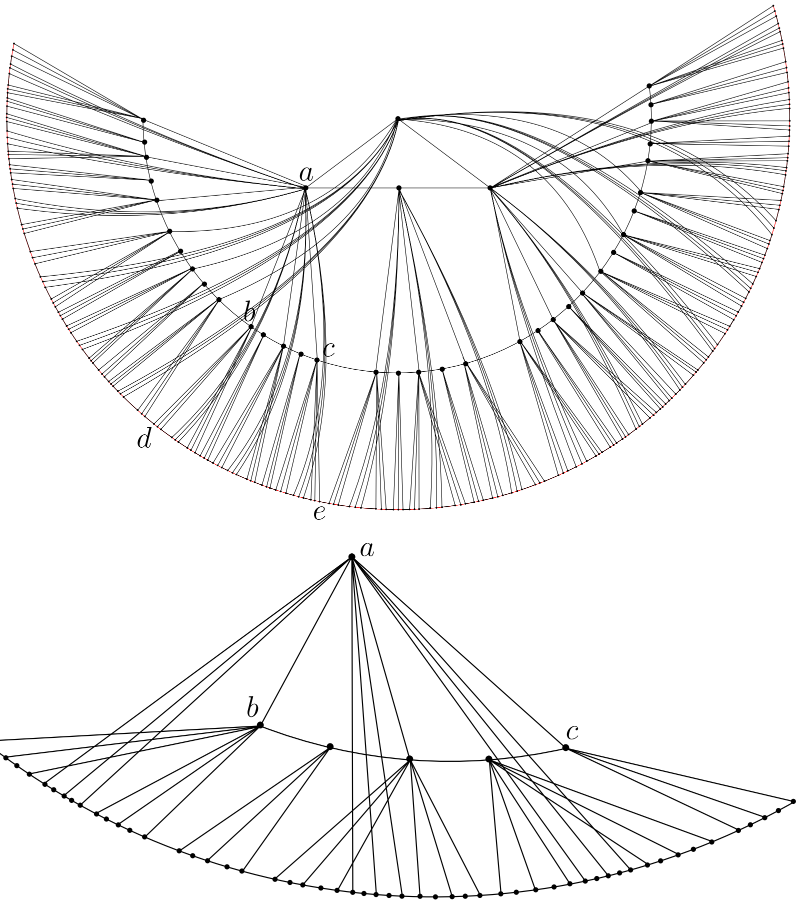


FIGURE 10: $G_{4,4}$ and a subgraph of $G_{4,4}$ (each red path in $G_{4,4}$ has length 4)

Now we will prove that for some integers k, l, the graph defined in Definition 3.5 contains K_l as a minor, as stated in Theorem 3.7. Beforehand, let us make some easy observation about the construction, as stated in the following lemma.

Lemma 3.6. *For every integers $l \geq 1$ and $k \geq 3$, there exists an (l, k)-graph.*

Proof. We prove by induction. For $l = 1$, the graph consists of a single vertex. For $l = 2$, it is a hole of length k, for some $k \geq 3$. Now for $k \geq 3$, graph $G_{l,k}$ is constructed from $G_{l-1,k}$ by adding a new path P_l, and every vertex in $G_{l-1,k}$ has neighbors in P_l, satisfying the properties of Definition 3.5.

□

Theorem 3.7. *For every integers $l \geq 1$ and $k \geq 4$, $G_{l,k}$ is $3PC(\cdot,\cdot)$-free, has girth k, and contains K_l as a minor.*

Proof. See Appendix B.2.

As a consequence of Lemma 3.4 and Theorem 3.7, we have the following result, concluding this section.

Corollary 3.7.1. *For every integers $l \geq 1$ and $k \geq 4$, $G_{l,k}$ is a $(\triangle, 3PC(\cdot,\cdot))$-free graph with girth k and treewidth at least l.*

Open question. What is the exact treewidth of K_4-free EHF graphs in terms of the number of vertices?

4 The Class of K_4-free EHF graphs (in progress)

We have seen in Section 2 that the class of EHF graphs has small treewidth when triangles are excluded. A natural question to ask is, what might happen when we allow the existence of triangle. As the first approach, we are going to study EHF graphs that do not contain a K_4. In the following section, we explain some characterizations of wheels contained in a graph of this class. We sometimes prove a lemma in a more general graph, namely K_4-free, odd-signable graphs which is a superclass of K_4-free EHF graphs.

4.1 Wheels in the class of K_4-free EHF graphs

We use the same definition as in Section 3.1 for multiple wheel and 2-wheel in K_4-free EHF graphs, except that we allow triangles but exclude even holes. Beforehand, let us define *basic 2-wheels of type-1 and of type-2*, as follows.

Definition 4.1. *Let G be a K_4-free, odd-signable graph. A 2-wheel $(H, \{u, v\})$ with $uv \notin E(G)$ is called basic if it has one of the following structure.*

(i) $|N_H(u)|, |N_H(v)| = 3$ s.t. $u_1, u_2, v_2, u_3, v_3, v_1$ appear in this order along H, with $u_1u_2, v_1v_3, v_2u_3 \in E(G)$, and possibly $u_1 = v_1$ or $u_3 = v_3$ (see Fig. 11 (a)).

(ii) $|N_H(u)| = 3$ with the two neighbors u_1, u_2 is contained in a sector V_1 of (H, v) s.t. $u_1u_2 \in E(G)$, and $u_3 = v_j$ for some $j \in \{3, 4, \cdots, n\}$ (see Fig. 11 (b)).

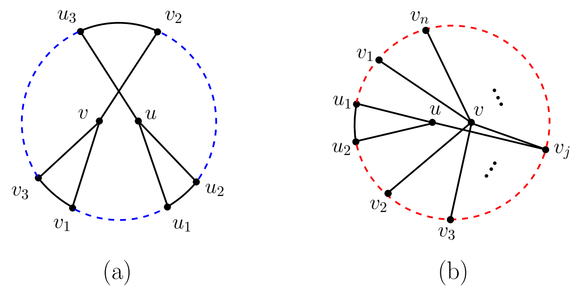


FIGURE 11: Basic wheels of type-1 (see Fig. (a)) and basic wheel of type-2 (see Fig. (b)), red and blue dashed edges are paths of odd and even (could be zero) length respectively

Now we are ready to describe the structures of 2-wheels in a K_4-free, EHF graph. Lemma 4.2 and Lemma 4.3 describes the case when the two centers of the 2-wheel are adjacent and non-adjacent respectively.

Lemma 4.2. *Let $(H, \{u, v\})$ be a 2-wheel that is contained in a K_4-free, EHF graph G. If u and v are non-adjacent, $(H, \{u, v\})$ is either basic or a nested 2-wheel.*

Proof. See Appendix B.3.

Lemma 4.3. *Let $(H, \{u, v\})$ be a 2-wheel that is contained in a K_4-free, odd-signable graph G. If u and v are adjacent, then they have at least one common neighbor in H.*

Proof. Let $\{v_1, \cdots, v_n\}$ be the set of neighbors of v in H. For a contradiction, assume u, v are adjacent but $N_H(u) \cap \{v_1, \cdots, v_n\} = \emptyset$. Then for every sector V_i of (H, v), $|N_{V_i}(u)|$ is even, since otherwise $V_i \cup \{u, v\}$ induces an even wheel. But then u has an even number of neighbors in H, so (H, u) is an even wheel, a contradiction.

□

4.2 Treewidth

In this section, we are going to explain our approach to extend the construction of a graph in the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs (see Definition 3.5) into a graph that does not contain a K_4 or an even hole. We aim to get insight of the treewidth of graphs in K_4-free EHF class. We propose a hypothesis regarding its treewidth.

Conjecture 4.4. *The class of K_4-free EHF graphs has unbounded treewidth.*

We now explain why we believe that the construction for $(\triangle, 3PC(\cdot,\cdot))$-free graph of arbitrarily large treewidth can be extended to K_4-free EHF graphs. Recall that the construction of $G_{l,k}$ that is described in Definition 3.5 defines a graph by iteratively building layers $P_1, \ldots, P_l$. Consider a layer P_i with $1 \leq i \leq l$. Note that by this construction, every vertex in $V(P_1) \cup \cdots \cup V(P_{i-1})$ has several neighbors in P_i.

First, each vertex of P_i has neighbors in P_{i+1}, and the vertices in P_i appear in the same order as their neighbor in P_{i+1}. This simple behavior cannot be kept in the EHF-construction. Indeed, two consecutive vertices in P_i would then eventually become adjacent centers of a wheel. By our construction in Definition 3.5, this case is impossible, because by Lemma 4.3, two adjacent centers of a multiple wheel in K_4-free EHF graph must have at least one common neighbor in the wheel. We therefore have to introduce triangles and local overlapping between neighbors of two adjacent vertices in layers $P_1, \cdots, P_l$. For example, see Fig. 12.

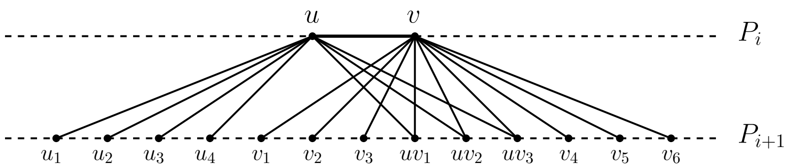


FIGURE 12: Overlapping of neighborhood of $uv \in E(P_i)$ in P_i

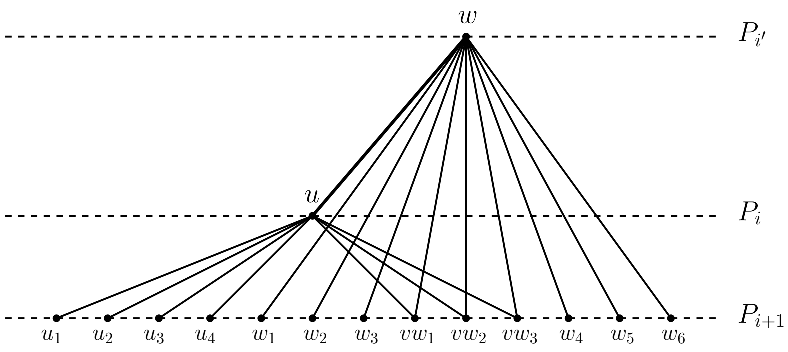


FIGURE 13: Overlapping of neighborhood of $v \in V(P_i)$ and its unique ancestor $w \in V(P_{i'})$ in P_{i+1}

Furthermore, we have to decide what to do with a vertex $v \in P_i$ that have an ancestor, say $w \in P_{i'}$ with $i' < i$. Again, the neighbors of v and w in P_{i+1} should overlap and yield triangle. See Fig. 13 to have an insight of it. Now there is a new problem. In P_{i+1}, there are vertices with two adjacent ancestors. The two ancestors could be in the same layer, or could be not. The fact is that in both cases, such a vertex and its two ancestors induce a triangle. Hence, we also have to handle the adjacency of such vertices when constructing layer P_{i+2}. This might be done as in Fig. 14.

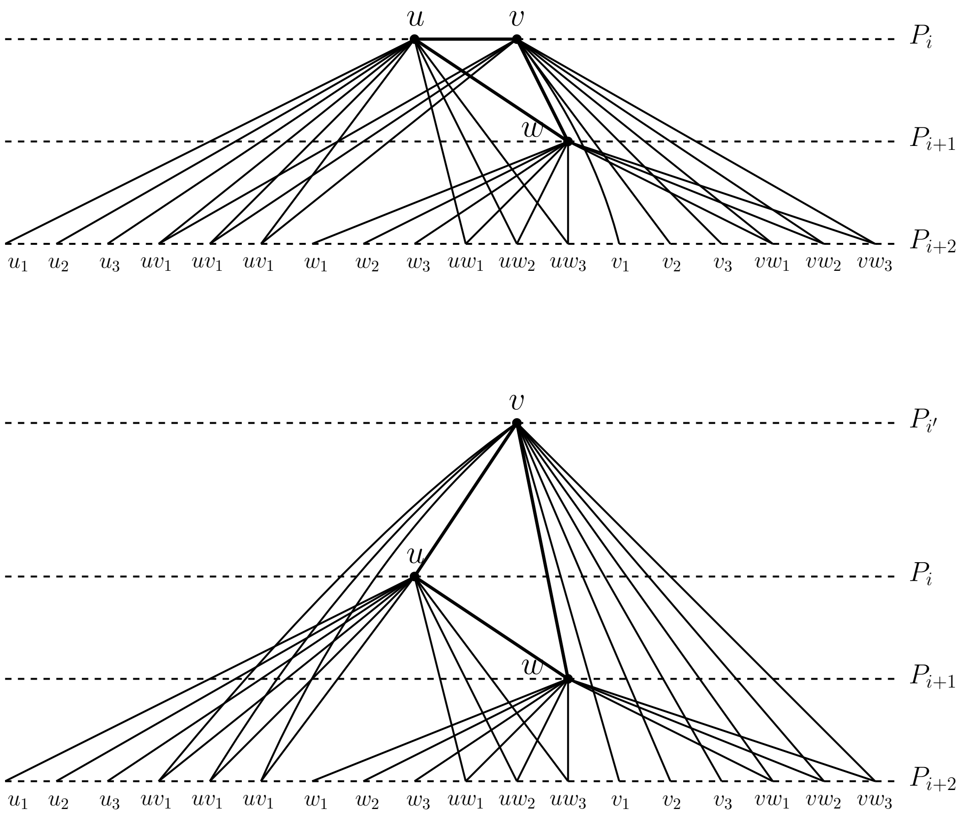


FIGURE 14: Overlapping of neighborhood of cycle (uvw) in P_{i+2} (top: when both ancestors of w are in the same layer, bottom: when they are not in the same layer)

It is tedious but not very difficult to prove that by carefully subdividing the path P_{t+1} (which is indeed a part of a wheel contained in our construction), all the cases above can be made even-hole-free (for instance, see Fig. 15 that is a subdivision of graph in Fig. 12). Putting these all together might lead to a construction of EHF graphs with no K_4 and it has arbitrarily large treewidth. An important feature of the construction is that every vertex has at most two ancestors, and if it has two, then those two ancestors are adjacent. Another important fact is that a vertex must have an odd number of neighbors in a layer, since otherwise, there would be an even wheel.

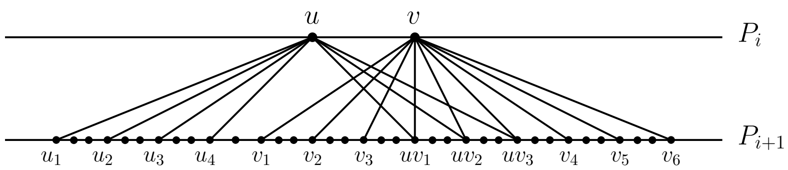


FIGURE 15: Example of subdivision of graph in Fig. 12

Furthermore, in constructing layer P_{i+2}, we need to arrange the neighborhood of vertices in P_j for $j < i+2$ that have neighbors in P_{i+1} in such a way that all the conditions above are satisfied. For instance, we could arrange the as the following (see Fig. 16)

The proof that the whole construction is even-hole-free could be made as follows. Consider for a contradiction a shortest even hole H in our new graph (by the above construction), and suppose w.l.o.g. that it contains vertices of P_l. If it contains a unique vertex in P_l, then there is a contradiction, since a vertex in P_l has at most two ancestors

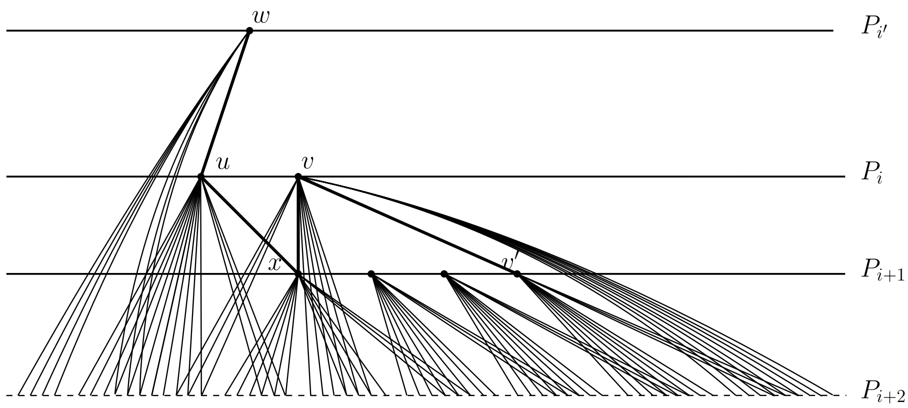


FIGURE 16: Putting together all the above conditions

(that are adjacent if there are exactly two of them). So, H in fact contains a subpath of length at least 1 in P_l. We hope that we can obtain a contradiction by replacing this subpath by shorter path, while preserving the parity.

In addition, we observe that graphs that are constructed in this way contains diamond (for instance, graphs in Fig. 14). This raises the following question: could it be that EHF graphs with no K_4 and diamond have bounded treewidth?

5 Conclusion and future research

In this report, we analyze some subclass of EHF graphs. Our first result shows that the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs and the class of $\triangle$-free EHF graphs are homeomorphic when cube is excluded, meaning that the structure of $\triangle$-free EHF graphs is not much simpler than the structure of $(\triangle, 3PC(\cdot,\cdot))$-free graphs which is its superclass. We question whether the same property may hold in general, namely the class of cube-free (or more general, square-free) EHF graphs.

Furthermore, we construct graphs which belong to the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs that contain K_l as a minor and having girth k for any integer $k, l \geq 1$. This leads to a conclusion that the class of $(\triangle, 3PC(\cdot,\cdot))$-free graphs have unbounded treewidth. However the exact treewidth for such graphs are still not yet computed, and it is also an interesting question. Indeed, we believe that the treewidth of the graphs is logarithmic in the number of vertices.

Moreover, in [22], the optimal coloring problem for this class are solved. In particular, graphs in this class are 3-colorable, and there exists an algorithm to color them in poly-time. The problem of finding maximum stable sets is still open. As been described in Section 4, we aim to extend this construction into K_4-free EHF graphs. We conjecture that the treewidth of this class of graphs is also unbounded.

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Appendices

A Background on Graph Theory

A graph $G(V, E)$ comprises of a set V of *vertices* and a set E of *edges*. It is *finite* if $|V|, |E|$ are finite. In an *undirected* graph, an edge is a non-ordered pair of vertices. An undirected graph without loops (i.e. an edge between a vertex and itself) or multiple edges (i.e. two or more edges joining a pair of vertices) is known as a *simple* graph. an edge connecting vertices u and v is denoted by uv, and vertices u and v are called end-nodes of the edge uv. Moreover, we say that u and v are *adjacent*, and that v is *incident* to e. The *degree* of a vertex is the number of edges incident to it. For a graph G, the *complement* of G, denoted as $\bar{G}$, is a graph $\bar{G}(\bar{V}, \bar{E})$ where $\bar{V} = V$ and for any $u, v \in \bar{V}$, $uv \in \bar{E}$ if and only if $uv \notin E$.

A *bipartite* graph is a graph whose set of vertices are decomposed into two disjoint sets which are called *bipartite components*, s.t. no two vertices within the same set are adjacent. It is called *complete bipartite* if every pair of vertices in the different components are adjacent. If the bipartite components are of size p and q respectively, the graph is denoted by $K_{p,q}$. A *complete* graph is a graph in which every pair of its vertices are adjacent. It is denoted by K_l if it contains l vertices. A *clique* in a graph is an induced subgraph that is isomorphic to a complete graph. The *clique number* of a graph, $\omega(G)$ is the number of vertices in a maximum clique in G (a clique K_l is maximum if no integer $l' > l$ s.t. $K_{l'}$ is a clique of G).

Skew partition of a graph is a partition of its vertices into two subsets, such that the induced subgraph formed by one of the two subsets is disconnected and the induced subgraph formed by the other subset is the complement of a disconnected graph (see Fig. 17 (d)). Here we also provide diagrams of graph decomposition by clique cutset, k-star cutset, and 2-join that are described in Section 1.1.2.

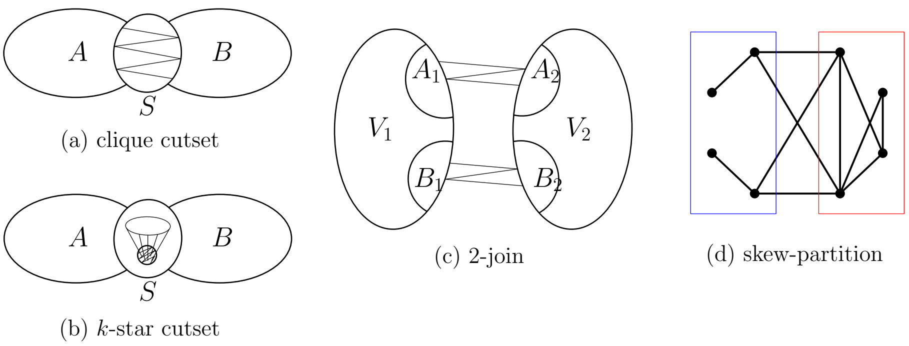


Figure 17: Diagrams of cutset decompositions

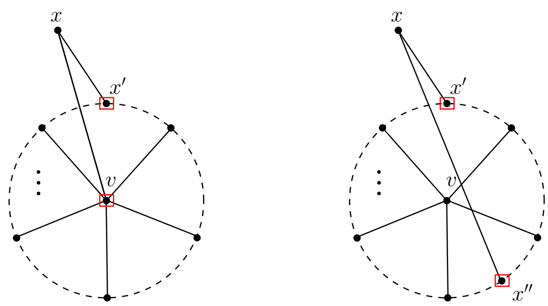
We give the definition of graph signing that is mentioned in Section 2. A graph G is a *signed graph* if the edges of G are given $0, 1$ labels. A *signing* of a graph G is assigning 0,1 weights to the edges of G. A subset of $E(G)$ has an odd (resp. even) *weight* if it contains an odd (resp. even) number of edges with sign 1.

B Proofs

B.1 Proof of Lemma 3.3

Lemma 3.3. *Let G be a $(\triangle, 3PC(\cdot,\cdot))$-free graph which does not contain a cube. Let (H, v) be a wheel contained in G, and x be a vertex in $G \backslash W$ that is strongly adjacent to (H, v). Then $H \cup \{v, x\}$ either induces a nested 2-wheel, or an alternated 2-wheel centered at $\{v, x\}$.*

Proof. First, suppose that x has exactly two neighbors in $W = (H, v)$. Assume $v \in N_W(x)$ (i.e. $vx \in E(G)$) then $N_H(x) \cap N_H(v) = \emptyset$ because G is $\triangle$-free. Let x' be the unique neighbor of x in $H \backslash N_H(v)$, then there exists a $3PC(v, x')$ (see Fig. 18 (a)). Otherwise assume $v \notin N_W(x)$, and let $N_W(x) = \{x', x''\} \subsetneq V(H)$. Note that $x'x'' \notin E(G)$ because G is $\triangle$-free. But then there exists a $3PC(x', x'')$ (see Fig. 18 (b)). So x has at least three neighbors in W. Now we consider two cases.



(a) (b)

FIGURE 18: If x has exactly two neighbors, then there exists $3PC(\cdot,\cdot)$

Case 1. $v \notin N_W(x)$ (i.e. $vx \notin E(G)$)

We will prove that $V(H) \cup \{v, x\}$ induces a nested 2-wheel. For a contradiction, suppose not. So some sector of (H, v) contains some neighbor of x but not all of them. First suppose that there exists a sector containing at least two neighbors of x, and w.l.o.g. let V_1 be such a sector. Let h_1, h_2 be the neighbors of v_1, v_2 in H that are not in V_1 respectively (recall that v_1 and v_2 are the end-nodes of V_1).

Claim B.0.1. $N_H(x) \subseteq V_1 \cup \{h_1, h_2\}$

Proof of Claim B.0.1. For a contradiction, assume that x^* is a neighbor of x in $H \backslash (V_1 \cup \{h_1, h_2\})$. But then there exists a $3PC(x, v)$ (see Fig. 19 (a)).

By the claim, $h_1 \in N_W(x)$ or $h_2 \in N_W(x)$. If $h_1 \in N_W(x)$, then $v_1 \notin N_W(x)$ because G is $\triangle$-free. But then there exists a $3PC(x, v_1)$ (see Fig. 19 (b)). The similar holds when $h_2 \in N_W(x)$. Hence, $N_W(x) \subseteq V_1$, a contradiction. So every v-sector contains at most one neighbor of x. Note that (H, x) is a wheel, because $|N_H(x)| \geq 3$. By symmetry, it implies that every x-sector contains at most one neighbor of v. Hence, for every sector V_i, there is a unique vertex x_i s.t. $x_i \in N_H(x)$, and for every sector X_i, there is a unique vertex v_{i+1} s.t. $v_{i+1} \in N_H(v)$, i.e. $|N_H(v)| = |N_H(x)|$ and $v_1, x_1, v_2, x_2, \cdots, v_n, x_n$ appear in this (counter-clockwise) order along H, where $n = |N_H(v)|$. Moreover, this implies that $N_H(v) \cap N_H(x) = \emptyset$. Note that, when $n \geq 4$ then there exists $3PC(u, v_1)$ (see Fig. 20 (a)). Otherwise, if $n = 3$, since G is cube-free,

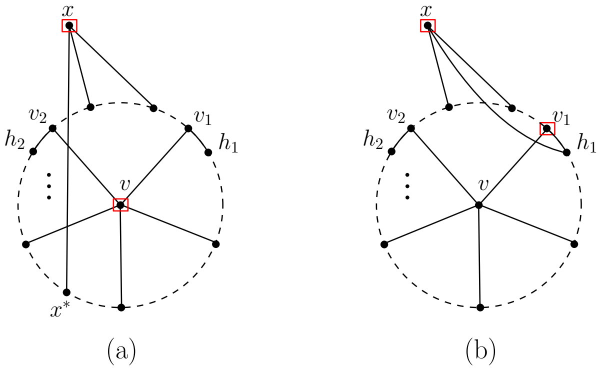


FIGURE 19: Proof of all neighbors of x are contained in one sector of (H, v)

then $G[H \cup \{v, x\}]$ is not a cube. Instead, it induces a cube where at least one of its edges is subdivided, and hence it contains a $3PC(\cdot, \cdot)$ (see Fig. 20 (b)), a contradiction.

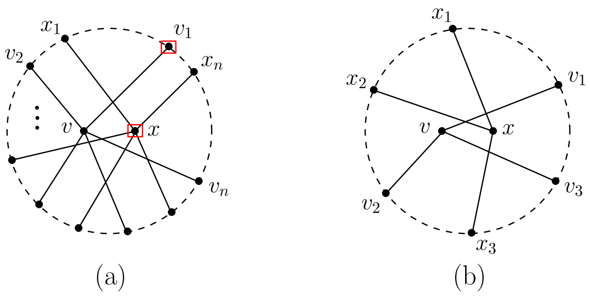


FIGURE 20: Case 1: every v-sector contains a unique one neighbor of x and every x-sector contains a unique neighbor of v

Case 2. $v \in N_W(x)$ (i.e. $vx \in E(G)$)

We will show that $V(H) \cup \{v, x\}$ induces either a nested 2-wheel or an alternated 2-wheel. We will show that every sector of (H, v) contains at least two neighbors of x. Suppose not, and w.l.o.g. let V_1 contains exactly one neighbor of x, say such neighbor is x'. Note that $x' \notin N_H(v)$ because G is $\triangle$-free, but then there exists a $3PC(v, x')$ (see Fig. 21 (a)). Hence every sector of (H, v) contains at least two neighbors of x, and by symmetry, every sector of (H, x) contains at least two neighbors of v. So $(H, \{v, x\})$ is an alternated 2-wheel (see Fig. 21 (b)). This completes the proof.

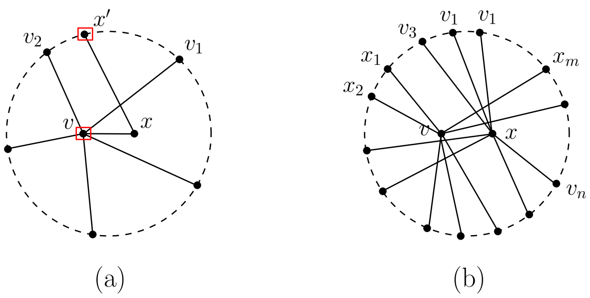


FIGURE 21: Case 2: $vx \in E(G)$

□

B.2 Proof of Theorem 3.7

Theorem 3.7. *For every integers $l \geq 1$ and $k \geq 4$, $G_{l,k}$ is $3PC(\cdot,\cdot)$-free, has girth k, and contains K_l as a minor.*

Proof. First, we will prove that $G_{l,k}$ has girth k. Let H be a hole contained in G. Consider the path of $G_{l,k}$ with largest index that contains some vertex of H. W.l.o.g. we may assume that P_l is this path. Let x be a vertex of $H \cap P_l$, and $P = uP_lv$ be the subpath of P_l that is included in H, containing x, and s.t. P is maximal, i.e. every vertex of P has index l and P cannot be made longer by adding some vertices of P_l. We may assume that $V(H) \cap V(P_i)$ is not a singleton, for otherwise, assume u is the unique vertex of H that is contained in P_l. Since every vertex in H has degree 2 in H and P_l contains a unique vertex of H, then u must have at least two ancestors, a contradiction. So P has length at least one, i.e. $u \neq v$, and both u and v have an ancestor. By construction, if u and v have a common ancestor, then $|V(P)| \geq k-1$ thus H has length at least k. If u and v have different ancestors, then $|V(P)| \geq k-2$ so H has length at least k.

Now we will show that $G_{l,k}$ does not contain any $3PC(\cdot,\cdot)$. For a contradiction, suppose it does, and let $\Theta = 3PC(u,v)$ be such a theta with the fewest number of vertices. Let x be a vertex with largest index in Θ. W.l.o.g. we may assume that x has index l, because all vertices of $G_{l,k}$ with index larger than the index of x can be deleted. Let P be the maximal subpath of P_l that contains x and that is included in Θ. If P has length zero, then its unique vertex must have two neighbors in Θ, that need to be ancestors, a contradiction. So, P has length at least 1, and each ends of P has an ancestor. We set $P = v_{l,j}P_lv_{l,j'}$, with $j < j'$.

Claim B.0.1. *$u, v \notin P_l$, in particular, every vertex of $\Theta \cap P$ has degree 2 in Θ.*

Proof of Claim B.0.1. For a contradiction, suppose $u \in V(P_l)$. Since $d_\Theta(u) = 3$ and $d_{P_l}(u) \leq 2$, then u has an ancestor. Let $v_{i,j}$ be its ancestor for some $1 \leq i \leq l-1$. By construction, $v_{i,j}$ has at least three neighbors in P_l, say $v_{l,a}, v_{l,b}, v_{l,c}$ with $a < b < c$, s.t. every vertex in $v_{l,a}P_lv_{l,c}\backslash\{v_{l,b}\}$ has degree 2 in $G_{l,k}$, and $u \in \{v_{l,a}, v_{l,b}, v_{l,c}\}$ (see Fig. 22). Since $d_{G_{l,k}}(u) = 3$, then it must be that $v_{i',j} \in V(\Theta)$, and a vertex of $\{v_{l,a}, v_{l,b}, v_{l,c}\}\backslash\{u\}$ is also contained in Θ. But then either $v_{i,j}v_{l,a}P_lv_{l,b}$ or $v_{i,j}v_{l,b}P_lv_{l,c}$ is a hole of Θ. W.l.o.g. suppose $v_{i,j}v_{l,a}P_lv_{l,b}$ is a hole in Θ, which yields that $\{u,v\} = \{v_{l,a}, v_{l,b}\}$ (because every vertex of $v_{l,a}P_lv_{l,b}$ has degree 2 in $G_{l,k}$). So $v_{l,a}P_lv_{l,b}$ and $v_{l,a}v_{i,j}v_{l,b}$ are two paths of Θ. Since $d_{G_{l,k}}(v_{l,b}) = 3$, then the third path of Θ must contain $v_{l,c}$, and hence $v_{i,j}v_{l,c}$ is a chord of Θ, a contradiction. This proves the claim.

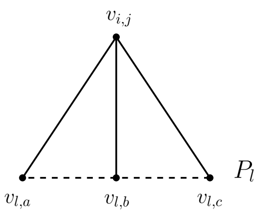


FIGURE 22: Claim B.0.1

Note that $v_{l,j}$ and $v_{l,j'}$ both have ancestor, say $v_{i,r}$ and $v_{i',r'}$ respectively with $1 \leq i, i' \leq l$ and $r < r'$ that are also in Θ. Moreover, it must be that $v_{l,j}$ and $v_{l,j'}$ are the unique neighbors of $v_{i,r}$ and $v_{i',r'}$ respectively in P, for otherwise a vertex of P has degree 3 in Θ, a contradiction to Claim B.0.1.

Claim B.0.2. *$v_{i,r} \neq v_{i',r'}$, $v_{i,r}v_{i',r'} \notin E(G_{l,k})$ and some internal vertex of P has an ancestor.*

Proof of Claim B.0.2. First, suppose that every internal vertex of P has degree 2 in $G_{l,k}$. By construction (see properties 7 and 8 of Definition 3.5), either $v_{i,r} = v_{i',r'} \in V(P_{l-1})$ or $v_{i,r}$ and $v_{i',r'}$ are adjacent. Hence, $v_{i,r}v_{l,j}Pv_{l,j'}v_{i',r'}$ is a hole of Θ, so the hole must contain both u and v. By Claim 1, $u, v \notin V(P_l)$, so $u, v \in \{v_{i,r}, v_{i',r'}\}$. But this is not possible as $v_{i,r} = v_{i',r'}$ or $v_{i,r}v_{i',r'} \in E(G_{l,k})$. This proves the claim.

We now set $P' = v_{i,r}v_{l,j}P_l v_{l,j'}v_{i',r'}$ (see Fig. 23).

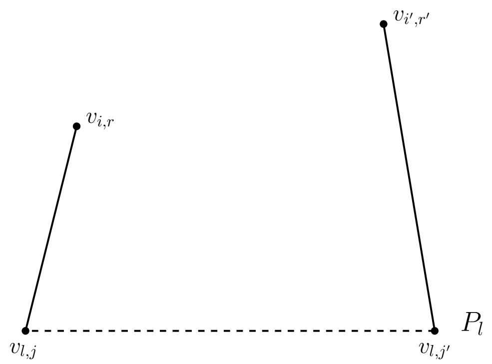


FIGURE 23: Path P'

Claim B.0.3. *$\mathring{P}$ does not contain any vertex with a type-0 ancestor with index $l-1$*

Proof of Claim B.0.3. For a contradiction let $v_{l-1,t}$ be a type-0 vertex with index $l-1$ that has children in the interior of P. Note that $v_{l-1,t} \notin V(\Theta)$ since $v_{l-1,t}$ has three neighbors in P (see Fig. 24). Now, consider a shortest path Q from $v_{i,r}$ to $v_{i',r'}$ in $G_{l,k}[V(P') \cup \{v_{l-1,t}\}]$. Note that Q is shorter than P', because it goes through $v_{l-1,t}$. So, P' can be substituted by Q in Θ, which provides a smaller $3PC(u, v)$, a contradiction to the minimality of Θ. This proves the claim.

By Claim B.0.2, some internal vertex of P has an ancestor. Let v_{i^*,r^*} be such an ancestor with i^* maximal. If $i^* \neq l-1$, then, by the construction of $G_{l,k}$, v_{i^*,r^*} has a neighbor w in P_{l-1}, and by maximality of i^*, w has no neighbor in the interior of P. Hence, by the construction of $G_{l,k}$, $w = v_{i,r} = v_{i',r'}$ (see Fig. 25), a contradiction. So, $i^* = l-1$. Note that $v_{l-1,r^*} \neq v_{i,r}$ and $v_{l-1,r^*} \neq v_{i',r'}$ because $v_{i,r}$ and $v_{i',r'}$ both have a unique neighbor in P. By Claim 2, v_{l-1,r^*} has an ancestor $v_{i^{**},r^{**}}$.

Suppose first that $v_{i^{**},r^{**}} \neq v_{i,r}$ and $v_{i^{**},r^{**}} \neq v_{i',r'}$. The neighbors of v_{l-1,r^*} along P_{l-1} are type-0 vertices, so by Claim 2, they have no neighbor in the interior of P. It follows that these two neighbors are $v_{i,r}$ and $v_{i',r'}$. Note that $v_{i^{**},r^{**}}$ has three neighbors in P, so $v_{i^{**},r^{**}} \notin V(\Theta)$. It follows that we obtain a shorter $3PC(u, v)$ by replacing P' with $R = v_{i,r}v_{l-1,r^*}v_{i',r'}$, a contradiction (see Fig. 26). Hence, $v_{i^{**},r^{**}} = v_{i,r}$ or $v_{i^{**},r^{**}} = v_{i',r'}$.

Up to symmetry, we suppose that $v_{i^{**},r^{**}} = v_{i',r'}$. Vertex v_{l-1,r^*-1} has type-0. So, by Claim B.0.3, it has no neighbor in the interior of P. Since it has neighbors in P, we have $v_{l-1,r^*-1} = v_{i,r}$. Let $v_{l,a}$, $v_{l,b}$, $v_{l,c}$, $v_{l,a'}$, $v_{l,b'}$, $v_{l,c'}$ be the six neighbors of v_{l-1,r^*} in P_l with $a < b < c < a' < b' < c'$ (see Fig. 27). Note that, $v_{l,a}, v_{l,b}, v_{l,c} \in V(P)$ and $v_{l,a'}, v_{l,b'}, v_{l,c'} \notin V(P)$.

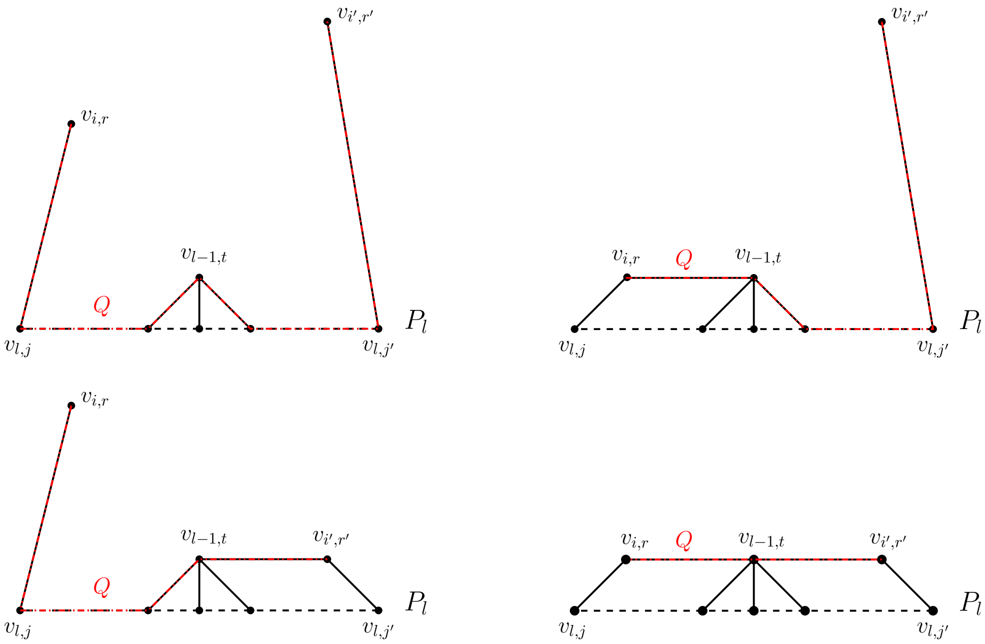


FIGURE 24: Four possible cases of replacing P' (see Fig. 23) with Q (red path) that yields a shorter $3PC(u, v)$

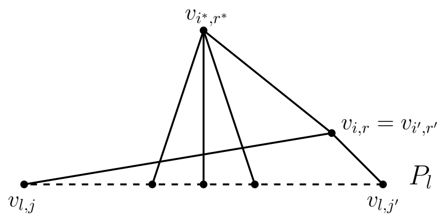


FIGURE 25: If $i^* \neq l - 1$ then $w = v_{i,r} = v_{i',r'}$

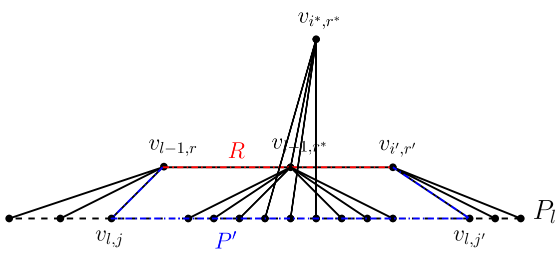


FIGURE 26: Replacing P' with R yields a shorter $3PC(u, v)$

If $\{v_{l,a'}, v_{l,b'}, v_{l,c'}\} \cap V(\Theta) = \emptyset$, then we obtain a shorter $3PC(u, v)$ by replacing P' with $v_{l-1,r}v_{l-1,r^*}v_{i',r'}$, a contradiction. So, $\{v_{l,a'}, v_{l,b'}, v_{l,c'}\} \cap V(\Theta) \neq \emptyset$. Let $v_{l,j''}$ be the neighbor of $v_{i',r'}$ with j'' maximal. Since $v_{l-1,r^*} \notin V(\Theta)$, $V(v_{i',r'}v_{l,j''}P_l v_{l,c'}) \subseteq V(\Theta)$. If $v_{i',r'} \notin \{u, v\}$, then by replacing $v_{l-1,r}P'v_{i',r'}v_{l,j''}P_l v_{l,c'}$ with $v_{l-1,r}v_{l-1,r*}v_{l,c'}$, we obtain a shorter $3PC(u, v)$, a contradiction. So, $v_{i',r'} \in \{u, v\}$, and w.l.o.g. we assume $v_{i',r'} = u$. If $v \neq v_{i-1,r}$, then by replacing $P'' = v_{l-1,r}P'v_{i',r'}v_{l,j''}P_l v_{l,c'}$ with $Q'' = G_{l,k}[\{v_{l-1,r}, v_{l-1,r^*}, v_{i',r'}, v_{l,c'}\}]$ (see Fig. 28), we obtain $3PC(v_{l-1,r^*}, v)$, which is shorter, a contradiction. So, $v = v_{l-1,r}$.

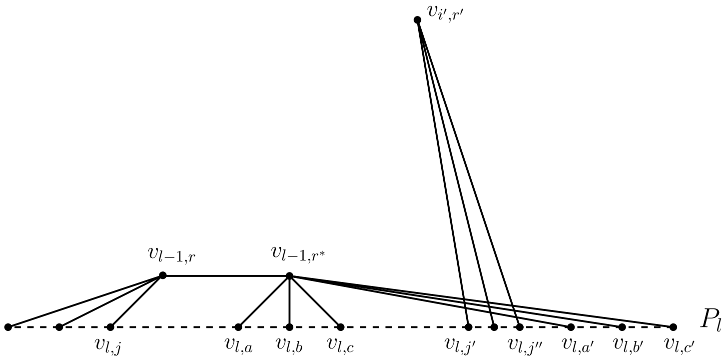


FIGURE 27

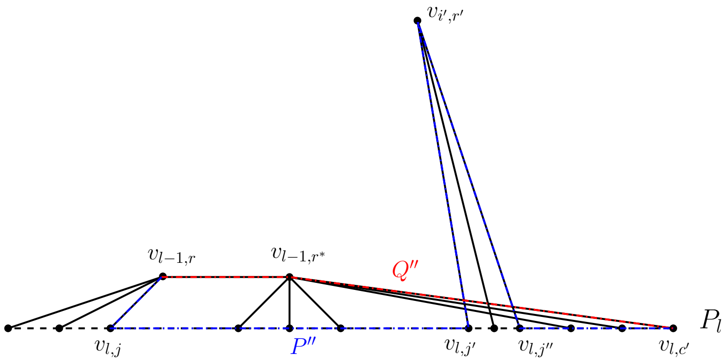


FIGURE 28: Replacing P'' with Q'' gives $3PC(v_{l-1,r^*}, v)$ shorter than $3PC(u, v)$

Since $v_{l-1,r}$ has type-0, we must have $v_{l-1,r-1} \in \Theta$. Let v_{l,a^*}, v_{l,b^*}, v_{l,c^*} be the three neighbors of $v_{l-1,r}$ in P_l, with $a^* < b^* < c^*$ (so that $c^* = j$). Since $v_{l-1,r}$ has type-0, $v_{l,a^*} \in \Theta$. Also, $v_{l,c^{**}} \in \Theta$, where $v_{l,c^{**}}$ is the neighbor of $v_{l-1,r-1}$ in P_l with c^{**} maximum (see Fig. 29). Hence, $v_{l-1,r-1}v_{l-1,r}v_{l,a*}P_l v_{l,c^{**}}v_{l-1,r-1}$ is a hole in Θ, a contradiction since u cannot be in that hole. Hence, $G_{l,k}$ is $3PC(\cdot,\cdot)$-free.

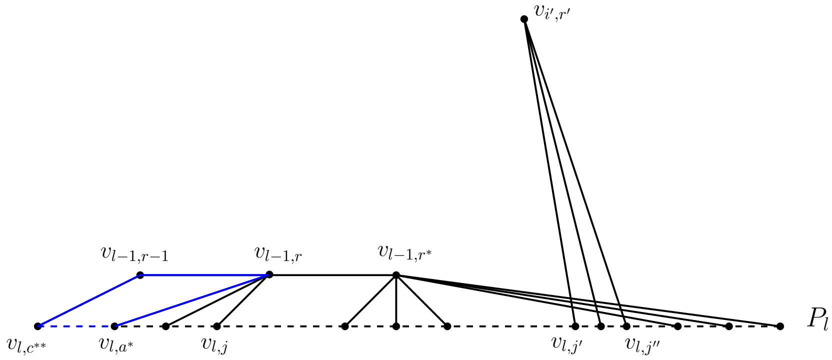


FIGURE 29: $v_{l-1,r-1}v_{l-1,r}v_{l,a*}P_l v_{l,c^{**}}v_{l-1,r-1}$ is a hole in Θ

So $G_{l,k}$ is $3PC(\cdot,\cdot)$-free and it has girth k. We now show that it contains K_l as a minor. Note that property 4 of Definition 3.5 guarantees that for $1 \leq i \leq l-1$, every vertex of index i has neighbors in $P_{i+1}, P_{i+2}, \cdots, P_l$. To get a K_l minor, then for every $1 \leq i \leq l$, contract the path P_i into a vertex. This gives us a graph on l vertices where any pair of vertices are adjacent.

□

B.3 Proof of Lemma 4.2

Beforehand, let us make an observation of some simple properties of a 2-wheel in the class of K_4-free EHF graphs.

Lemma B.1. *Let $(H, \{u, v\})$ be a 2-wheel that is contained in G. If u and v are non-adjacent, then u and v have at most two common neighbors, and if it is two, then they are adjacent.*

Proof. Suppose that u and v have three common neighbors in H. Since H is a hole, then at least two of the common neighbors are non-adjacent. But then those two common neighbors together with u and v induce a C_4. In the case where u and v have exactly two common neighbors but non-adjacent, they also induce a C_4. This proves the lemma.

□

Lemma B.2. *Let v be a vertex that is strongly adjacent to a hole H in G. Then one of the following holds*

(i) It has exactly two neighbors that are adjacent in H

(ii) It has an odd number of neighbors in H

Proof. First assume that v has exactly two neighbors in H, say $N_H(v) = \{v_1, v_2\}$. For a contradiction, suppose $v_1v_2 \notin E(G)$. But then, there exists $3PC(v_1, v_2)$ with $P_1 = v_1vv_2$, $P_2 = v_1H_1v_2$, $P_3 = v_1H_2v_2$ where H_1, H_2 are two subpaths of H connecting v_1 and v_2. Furthermore, if v has at least three neighbors in H, then (H, v) is a wheel. Since G is odd-signable, then $|N_H(v)|$ must be even, since otherwise (H, v) is an even wheel. This proves the lemma.

□

Lemma 4.2. *Let $(H, \{u, v\})$ be a 2-wheel that is contained in a K_4-free, EHF graph G. If u and v are non-adjacent, $(H, \{u, v\})$ is either basic or a nested 2-wheel.*

Proof. Let $N_H(v) = \{v_1, \cdots, v_n\}$ and $N_H(u) = \{u_1, \cdots, u_m\}$, with $n, m \geq 3$ and odd. For a contradiction, suppose that $(H, \{u, v\})$ is not a nested 2-wheel. Hence, some but not all neighbors of u are contained in a sector, w.l.o.g. V_1. Let $N_{V_1}(u) = \{u_1, \cdots, u_k\}$ with $1 \leq k < n$. We consider two cases.

Case 1. v has at least five neighbors in H (i.e. $n \geq 5$)

Case 1.1. $k \geq 3$

Let V_i with $i \neq 1$ be a sector containing some vertices of $\{u_{k+1}, \cdots, u_n\}$. Note that k is odd because the graph does not contain an even wheel. So there are at least two neighbors of u that are not in V_1. Suppose h_1 and h_2 are the neighbors of v_1 and v_2 respectively, that are in H but not in V_1. We claim that $N_H(u) \subseteq V_1 \cup \{h_1, h_2\}$. Since otherwise, there exists $3PC(u, v)$ (see Fig. 30 (a)). Note that both h_1 and h_2 are indeed neighbors of u. Furthermore, by Lemma B.1, it cannot be that both $u_1 = v_1$ and $u_k = v_2$. Moreover, if both $u_1 \neq v_1$ and $u_k \neq v_2$, then there exists $3PC(v_1, u)$ or $3PC(v_2, u)$ unless both $v_n = h_1$ and $v_3 = h_2$ (see Fig. 30 (b)). But the last case ($v_n = h_1$ and $v_3 = h_2$) is not possible by Lemma B.1. So either $u_1 = v_1$ and $u_k \neq v_2$, or $u_1 \neq v_1$ and $u_k = v_2$. W.l.o.g. suppose that $u_1 = v_1$ and $u_k \neq v_2$. Note that it cannot be that $v_3 = h_2$, since otherwise, Lemma B.1 is contradicted. But then there exists $3PC(v_2, u)$ (see Fig. 30 (c)), a contradiction.

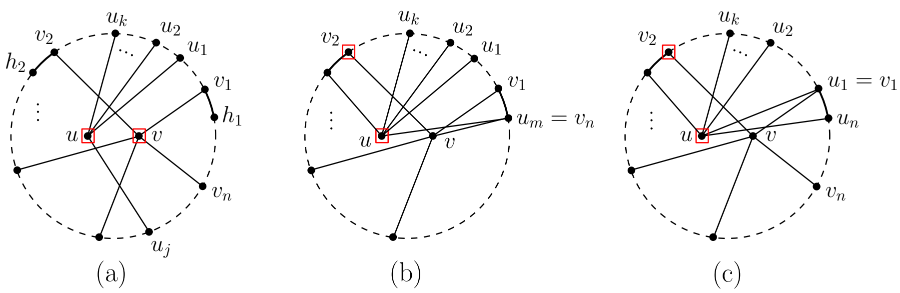


FIGURE 30: Case 1.1

Case 1.2. $k = 2$

Applying Lemma B.2 to the hole that is induced by $V_1 \cup \{v\}$, we obtain $u_1u_2 \in E(G)$. First suppose that $m = 3$. We claim that $u_3 \in \{v_3, \cdots, v_n\}$. Since otherwise, there exists $3PC(u_3, v)$ (see Fig. 31 (a)). Hence, $(H, \{u, v\})$ induces a basic 2-wheel of type-2 (see Fig. 31 (b)). Now suppose that $m \geq 5$. Note that, if there exists a u-sector containing at least three neighbors of v then we are in a similar case to Case 1.1. So, every v-sector contains at most two neighbors of u, and every u-sector contains at most two neighbors of v. We claim that if a v-sector containing a single neighbor of u then such vertex must be one of the end-nodes of the sector. For otherwise, suppose $V_i \neq V_1$ and $u' \in N_H(u) \cap V_i$ be s.t. the statement is contradicted. Then there exists $3PC(u', v)$ (see Fig. 31 (c)). By Lemma B.1, u and v have exactly one common neighbor, or two common neighbors that are adjacent. In fact, the latter case is not possible. To see this, suppose that V_i is a v-sector whose end-nodes are adjacent and they are neighbors of u. Hence, other v-sectors must contain exactly two neighbors of u in its interior. But then (H, u) is an even wheel, a contradiction.

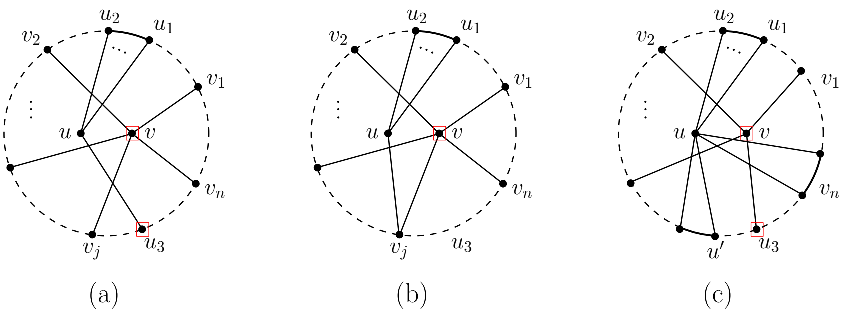


FIGURE 31: Case 1.2

So, except on one sector where u and v have a common neighbor, every v-sector contains either none or two neighbors of u, and every u-sector contains either none or two neighbors of v. W.l.o.g. we may assume that v_n are the common neighbor of u and v (i.e. $u_n = v_n$). By observing the parity of every path in H connecting two vertices which are neighbors of u or v, we obtain that $C = uu_{n-1}V_{n-2}v_{n-1}vv_2V_1u_2u$ is an even hole. To be more precise, both of $|u_nHv_1|$ and $|u_1Hu_2|$ are odd. Since $|u_nHu_1|$ is also odd then $|v_1Hu_1|$ is even. It turns out that $|u_2Hv_2|$ is even because $|v_1Hv_2|$ is odd. But then the hole $C = uv_nvv_2Hu_2u$ has even length (see Fig. 32), a contradiction.

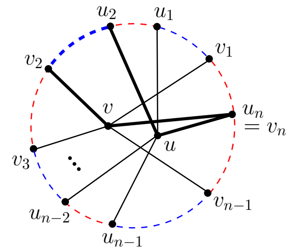


FIGURE 32: Case 1.2: $C = uv_nvv_2Hu_2u$ is an even hole in $(H, \{u, v\})$ (the dashed red and blue edges are paths of odd and even length respectively)

Case 1.3. $k = 1$

It turns out that every v-sector contains at most one neighbor of u. First suppose that $m = 3$. By Lemma B.1, it cannot be all of $N_H(u)$ are the neighbors of v, i.e. there exists a v-sector whose internal vertices contain a single neighbor of u. W.l.o.g. let V_1 be such a sector. We claim that $u_2 = h_2$ and $u_3 = h_1$ where h_1 and h_2 are the neighbors of v_1 and v_2 respectively in H, but are not in V_1. For otherwise, there exists $3PC(u_1, v)$ (see Fig. 33 (a)). So $u_3, v_1u_1v_2u_2$ appear in this (counter-clockwise) order along H with $u_2v_2, u_3v_1 \in E(G)$, and possibly $v_n = u_3$ or $v_3 = u_2$ (see Fig. 33 (b)). Note that by Lemma B.1, it cannot be that both $v_n = u_3$ and $v_3 = u_2$. W.l.o.g. suppose that $v_n \neq u_3$. But then there exists $3PC(v_1, u)$ (see Fig. 33 (b)), a contradiction.

Now suppose that $m \geq 5$. Note that, if there exists a u-sector containing at least two neighbors of v, then we are done (it is similar to Case 1.1. or Case 1.2.). So every u-sector contains a single neighbor of v. This implies that $N_H(v)$ and $N_H(u)$ alternate in H, and hence u and v have no common neighbor. So every v-sector contains a unique neighbor of u in its interior. But then there exists $3PC(u, v)$ (see Fig. 33 (c)). Hence, $(H, \{u, v\})$ is a nested 2-wheel. This completes the proof of Case 1.

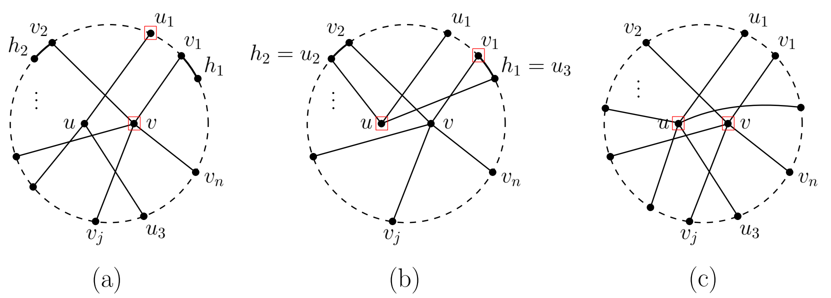


FIGURE 33: Case 1.3

Case 2. v has exactly three neighbors in H (i.e. $n = 3$)

We may assume that $m = 3$, since for $m \geq 5$, we are in the symmetric case of Case 1. So $m = n = 3$, and since $k < m$, then $k = 2$ or $k = 1$.

Case 2.1. $k = 2$

Applying Lemma B.2 to the hole that is induced by $V_1 \cup \{v\}$, we obtain $u_1u_2 \in E(G)$. If $u_3 = v_3$ then $(H, \{u, v\})$ is a basic 2-wheel of type-2 (see Fig. 34 (a)). Otherwise, by symmetry, assume that $u_3 \in \mathring{V}_2$. Note that by Lemma B.2, it turns out that $v_1v_3 \in E(G)$. If $u_3v_2 \notin E(G)$ then there exists $3PC(u_3, v_2)$ (see Fig. 34 (b)), and if $u_3v_2 \in E(G)$ then $(H, \{u, v\})$ induces a basic 2-wheel of type-1 (see Fig. 34 (c)).

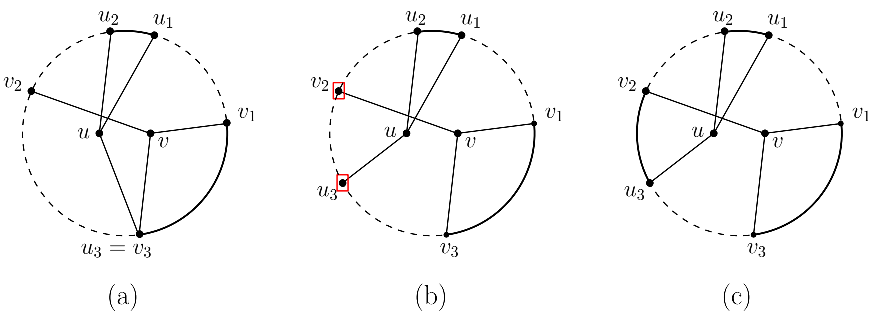


Figure 34: Case 2.1

Case 2.2. $k = 1$

It turns out that every v-sector contains a unique neighbor of u, and by symmetry, every u-sector contains a unique neighbor of v. Hence, $V(H) \cup \{u, v\}$ either induces a cube (see Fig. 35 (a)) and thus contains a C_4, or it is a cube where at least one of its edges is subdivided and thus contains a $3PC(\cdot, \cdot)$ (see Fig. 35 (b)). In both cases, we obtain a contradiction. This completes the proof of Lemma 4.2.

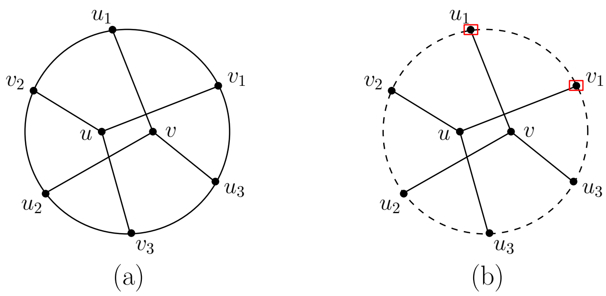


Figure 35: Case 2.2

□